Math 304 (Spring 2015) - Homework 6

Problem 1.

Find the transition matrix from the basis $\{u_1 = \begin{pmatrix}1\\1\end{pmatrix}, u_2 = \begin{pmatrix}1\\-1\end{pmatrix}\}$ to the standard basis $\{e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}\}$

Solution: The transition matrix is

$$U = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Problem 2.

Let $\mathbb{P}_2$ be the vector space of polynomials with degree ≤ 2. We know that $\{1, x, (x+1)^2\}$ is a basis of $\mathbb{P}_2$. Find the coordinate vector of the polynomials $p(x) = x^2 - 1$ with respect to the basis $\{1, x, (x+1)^2\}$.

Solution:

$$x^2 - 1 = a \cdot 1 + b \cdot x + c \cdot (x+1)^2$$

Solve for a, b and c.

$$\begin{cases} a + c = -1 \\ b + 2c = 0 \\ c = 1 \end{cases}$$

So the coordinate vector is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

Problem 3.

Given the matrix

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix}$$

(a) Find a basis of the row space of A and use it to determine the rank of A.

(b) Find a basis of the column space of A.

(c) Find a basis of the null space of A.

Solution:

(a) Row echelon form of A is

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So

$$u_1 = (1, 2, -1, -2)$$
$$u_2 = (0, 1, 0, -2/7)$$
$$u_1 = (0, 0, 1, 0)$$

form a basis of the row space of A. So the rank of A is 3.

(b) By looking at the pivotal entries of the echelon form of A, we see that

$$v_1 = \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

form a basis of the column space of A.

(c) All solutions are of the form

$$s \begin{pmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix}$$

So

$$w = \begin{pmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix}$$

forms a basis of $\ker A$.

Problem 4.

Determine whether the following mappings are linear transformations.

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ c \end{pmatrix}$$

(b) $L : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$L\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a^2+b^2 \\ c \end{pmatrix}$$

(c) Let $\mathbb{P}_3$ be the vector space of all polynomials with degree ≤ 3. The mapping $L : \mathbb{P}_3 \to \mathbb{P}_3$ by

$$L(p(x)) = p'(x)$$

where $p'(x)$ is the derivative of $p(x)$.

(d) $L : \mathbb{P}_2 \to \mathbb{P}_3$ by

$$L(p(x)) = x \cdot p(x)$$

(e) $L : \mathbb{P}_2 \to \mathbb{P}_3$ by

$$L(p(x)) = p(x) + x^2$$

Solution: For this question, I will skip the details.

(a) Yes.

(b) No.

(c) Yes.

(d) Yes.

(e) No.

Problem 5.

Find the matrix representations of the following linear transformations.

(a) Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ by

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{pmatrix}$$

Find the standard matrix representation of L.

(b) The vectors

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

form a basis of $\mathbb{R}^3$. Let L be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$L\begin{pmatrix}x_1\\x_2\end{pmatrix} = x_1v_1 + (x_2 + x_1)v_2 + (x_1 - x_2)v_3.$$

Find the matrix representation of L with respect to the bases $\{e_1, e_2\}$ (the standard basis of $\mathbb{R}^2$) and $\{v_1, v_2, v_3\}$.

(c) The vectors

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

form a basis of $\mathbb{R}^3$. Let L be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$L\begin{pmatrix}x_1\\x_2\end{pmatrix} = x_1v_1 + (x_2 + x_1)v_2 + (x_1 - x_2)v_3.$$

Find the matrix representation of L with respect to the standard bases.

Solution:

(a)

$$L(e_1) = \begin{pmatrix}2\\-1\\-1\end{pmatrix}, L(e_2) = \begin{pmatrix}-1\\2\\-1\end{pmatrix}, L(e_3) = \begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

The standard matrix representation is

$$\begin{pmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{pmatrix}$$

(b)

$$L(e_1) = v_1 + v_2 + v_3$$

so its coordinate vector with respect to $\{v_1, v_2, v_3\}$ is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Similarly, the coordinate vector of $L(e_2)$ is

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

So the matrix representation is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(c)

$$L(e_1) = v_1 + v_2 + v_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

so its coordinate vector with respect to the standard basis $\{e_1, e_2, v_3\}$ of $\mathbb{R}^3$ is

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Similarly, the coordinate vector of $L(e_2)$ is

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

So the matrix representation is

$$\begin{pmatrix} 2 & 1 \\ 2 & -1 \\ 2 & 0 \end{pmatrix}$$